A Modigliani-Miller Theorem for
Open-Market Operations

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Abstract

It is shown that if fiscal policy is held fixed in an appropriate way, then different government portfolios are irrelevant in precisely the sense in which different corporate liability structures are irrelevant under the Modigliani-Miller theorem. Holding fiscal policy fixed means turning back to the private sector via the tax system differences in net-interest-received by the government implied by different portfolios. Also examined are departures from the assumptions needed to get irrelevance. It is suggested that some of the usually asserted effects of open-market operations can be understood in terms of these departures.

A Modigliani-Miller Theorem for Open-Market Operations

Monetary policy determines the composition of net government

indebtedness or the path of the government's portfolio. Fiscal

policy, in particular, the size of the deficit on current account,

determines the path of net government indebtedness. In this paper I

will show that alternative paths of the government's portfolio

consistent with a single path of fiscal policy can be irrelevant in

precisely the sense in which the Modigliani-Miller theorem shows

that alternative corporate liability structures are irrelevant.

Irrelevance here means that both the equilibrium consumption allocation and

the path of the price level are independent of the path of the government's

portfolio. Roughly speaking, the irrelevance proposition I prove has

the following form: if there is an equilibrium with certain properties

for one path of portfolios for the government, then that equilibrium

is also an equilibrium for a large class of other paths of portfolios

for the government provided only that lump-sum taxes are adjusted in

an appropriate way. Appropriate means, among other things, that fiscal

policy is held constant.

I prove the irrelevance result for a limited class of environments:

models of two-period-lived, overlapping generations and of a single

consumption good that is storable via a constant-returns-to-scale,

stochastic technology. This class of models, described in section one,

is broad enough to include examples that establish the nonvacuousness

of the "if" clause of the irrelevance proposition. Nonvacuousness

requires that there be equilibria in which the private sector

voluntarily holds real capital and unbacked government liabilities,

liabilities that I call fiat money. The overlapping-generations

structure and risky real capital (storable consumption good) make

this possible.

In section two, I describe the conditions for a perfect-foresight

competitive equilibrium for the section one environments. The

irrelevance proposition is presented, remarked on, and proved in

section three. The proposition establishes conditions under which

the amount of the consumption good purchased by the government in the

open market for fiat money and stored by the government is irrelevant.

Complete markets in contingent claims play a prominent role just as

they do in the Stiglitz (1969) version of the Modigliani-Miller theorem.

In sections four and five, I consider by way of examples departures

from the assumptions of the irrelevance proposition. In the former,

I describe a departure that arises when the nonnegativity restriction

on private gross investments is binding. In the latter, I describe a

departure that arises when a legal restriction on minimum money holdings

is binding. These examples establish the necessity of the voluntarily-

diversified asset-holding assumptions of the irrelevance proposition.

They are also of interest because they offer a microeconomic interpre-

tation of the usual macroeconomic-model analysis of open-market operations.

Open-market operations in default-free government bonds are discussed

in section six. I argue that nonirrelevance depends on the presence of

"transaction-costs" asymmetries between the private sector and the govern-

ment. The nonirrelevance possibilities described in section four through

six seem to share two features. If two different paths of the government's portfolio necessarily imply different equilibria, then they also imply different paths of fiscal policy. Moreover, at least one of the paths can be viewed as running into barriers to private intermediation.

1. The Physical Environment

Time is discrete and there is a single good. At each date t, a new generation of $N(t)$ 2-period-lived individuals (generation t) appears. Each member h of generation t maximizes the expected value of $u^h(\ ,\ )$, where the first (second) argument is consumption of the good by h in the first (second) period of life and where u^h is strictly increasing, strictly concave and twice differentiable.

At each date t, there is a new aggregate endowment of $Y(t) > 0$ units of the consumption good. This good may be consumed or stored. If $K(t) \geq 0$ is the aggregate amount placed into storage at t, then $Y(t+1) + K(t)x(t+1)$ is the total amount available at $t+1$, where $x(t+1)$ is a random variable drawn independently from period to period from a discrete probability distribution: $x(t+1) = x_i > 0$ with probability f_i; $i=1,2,\ldots,I$. The I-element vector $(x_1,x_2,\ldots,x_I)$ will be denoted x. The value of $x(t+1)$ is observed after time t storage is determined and before generation $t+1$ appears. Note that $K(t)$ is the sum of nonnegative private storage $K^P(t)$ and nonnegative government storage $K^g(t)$.

The supply of fiat money is determined by the government. Changes in it do not require the expenditure of resources by the government and private storage of fiat money neither affects its physical properties nor requires the expenditure of resources.

2. The Market Scheme

I will describe the conditions for a perfect foresight competitive equilibrium in terms of time t markets for claims on time $t+1$ consumption in "state" $x(t+1) = x_i$. The members of generation t in their role as consumers demand such claims. Firms, owned by members of generation t in their role as producers, supply such claims by storing the consumption good and by storing fiat money. In general, the government announces a policy, including a lump-sum tax-transfer scheme, in terms of such claims.

The Consumer's Lifetime Choice Problem

The consumer choice problem of the young of generation t is described in terms of the following notation:

$(c_1^h(t), c_2^h(t))$ — The $(I+1)$ element consumption vector of member h of generation t where $c_1^h(t)$ is first-period consumption and $c_2^h(t) = (c_{21}^h(t), c_{22}^h(t), \ldots$
$\ldots, c_{2I}^h(t))$, $c_{2i}^h(t)$ being second-period consumption in "state" $x(t+1) = x_i$.

$(w_1^h(t), w_2^h(t))$ — the corresponding $(I+1)$ element endowment vector of member h of generation t, where
$w_2^h(t) = (w_{21}^h(t), w_{22}^h(t), \ldots, w_{2I}^h(t))$.

$s(t)$ — the I-element vector $(s_1(t), s_2(t), \ldots, s_I(t))$ where $s_i(t)$ is the price at time t of one unit of $t+1$ consumption in "state" $x(t+1) = x_i$ in units of time t consumption.

Later, it will be convenient to have a notation for the consumption allocation and endowment allocation of generation t: let $c(t)$ be the $N(t)(I+1)$ – element vector consisting of one $(c_1^h(t), c_2^h(t))$ vector for each member h of generation t and let $w(t)$ be the corresponding $N(t)(I+1)$ – element endowment vector.

All of this notation is meant to allow for possible dependence of, say, $s(t)$ on $x(t)$, $x(t-1)$ and so on. For any variable $\cdot(t)$, dependence on t is used to denote possible dependence on $x(t)$, $x(t-1)$,.... This is a convenient notation because the young of generation t make choices having observed $x(t)$ and all lagged x's.

Member h of generation t is assumed to choose a nonnegative vector $(c_1^h(t), c_2^h(t))$ to maximize $\Sigma_i f_i u^h[c_1^h(t), c_{2i}^h(t)]$ subject to

(1)
$$c_1^h(t) + s(t)c_2^h(t) \le w_1^h(t) + s(t)w_2^h(t)$$

where the vector multiplication is inner-product multiplication. For $s(t)$ and $(w_1^h(t), w_2^h(t))$ that imply a non-empty, bounded budget set, there is a unique maximizing vector $(c_1^h(t), c_2^h(t))$ given by the unique solution to (1) at equality and

(2)
$$f_i u_2^h[c_1^h(t), c_{2i}^h(t)] = s_i(t) \sum_{j=1}^{I} f_j u_1^h[c_1^h(t), c_{2j}^h(t)];$$

$$i=1,2,\ldots,I$$

This is all that need be said about consumer demand.

The Choice Problem of Firms

In their role as producers, members of generation t may enter one or both of two lines of business at time t: storing the consumption good or storing money. In each line, any producer maximizes profit as a price-taker with regard to s(t) and the time t and time t+1 prices of money.

Profit in terms of time t consumption from storing k $\geq$ 0 units of the consumption good is s(t)xk - k. Since this is linear in k, the condition that storage be finite in any equilibrium implies as an equilibrium condition

$$(3) \qquad s(t)x \leq 1$$

a condition that must hold with equality if total private storage, $K^P(t)$, is positive.

If p(t) is the price of a unit of money at time t in units of time t consumption and p(t+1) is the price of a unit of money at time t+1 in terms of time t+1 consumption (an I-element vector as of time t), then profit in terms of time t consumption from storing m $\geq$ 0 units of fiat money is s(t)p(t+1)m - p(t)m. Since this is linear in m, finiteness of the supply of money implies that prices in any competitive equilibrium satisfy

$$(4) \qquad s(t)p(t+1) = p(t)$$

We may write equality here because if firms store no money, then demand falls short of supply and p(t) = 0.

Government Policy Rules

Government policy is a specification at time $t=1$ after $x(1)$ has been observed of paths, possibly contingent, for government consumption at t, $G(t) \geq 0$; the endowment vector for generation t, $w(t)$; government storage at t, $K^g(t)$; and the money supply at t, $M(t) \geq 0$. For $t>1$ and each $x(t)$ in x, these are chosen subject to

$$(5) \qquad K^g(t) + G(t) = T(t) + K^g(t-1)x(t) + p(t)[M(t)-M(t-1)]$$

Here $T(t)$, total lump-sum taxes minus transfers at t, is defined by[1]

$$(6) \qquad T(t) = Y(t) - \Sigma_h w_1^h(t) - \Sigma_h w_{2i}^h(t-1)$$

and $M(0)$, $w(0)$ (the endowment of the old at $t=1$), and $K^g(0)$ are assumed given as initial conditions. (The summations over h are over the members of generation t and $t-1$, respectively, a convention that will be used throughout.)

Perfect Foresight Competitive Equilibrium

The question of foresight arises with regard to $p(t+1)$ in (4) and with regard to $w_2^h(t)$ in (1). Perfect foresight requires that the i-th element of $p(t+1)$ in (4) equal the equilibrium price of money at $t+1$ in "state" $x(t+1) = x_i$ and that the $w_2^h(t)$ vector on the basis of which h chooses at t be realized at $t+1$. Put formally, then, for specified government policy consisting of a possibly contingent sequence $(G(t), w(t), K^g(t))$ defined for

$t \geq 1$, a perfect foresight competitive equilibrium consists of nonnegative sequences $c(t-1)$, $s(t)$, $K(t) \geq K^g(t)$, $p(t)$ and $M(t)$ that for all $t \geq 1$ satisfy (1) at equality and (2) for each h, (3) – (6) and

$$(7) \qquad \Sigma_h (c_{2i}^h(t) - w_{2i}^h(t)) = K^p(t)x_i + p_i(t+1)M(t)$$

for each i. The LHS of (7) is the aggregate excess demand of consumers for consumption at $t+1$ in state $x(t+1) = x_i$, while the RHS is the supply of such consumption by firms in that state.

An example

Here is an example of an economy with a diversified equilibrium.

Physical environment: For all t, $N(t) = N$, $Y(t) = yN > 0$ and $u^h(z_1, z_2) = \ln(z_1) + \ln(z_2)$ for all h; $x = (x_1, x_2) = (0.5, 2.0)$ and $f_1 = f_2 = 0.5$.

Policy: For all $t \geq 1$, $G(t) = K^g(t) = 0$ and $w_1^h(t) = y$, $w_{2i}^h(t) = 0$ for all i and h.

Equilibrium: For all $t \geq 1$, $(s_1(t), s_2(t)) = (2/3, 1/3)$, $K(t)/N = y/4$, $M(t) = M(1)$, $p(t)M(t)/N = y/4$ and for all h, $(c_1^h(t), c_{21}^h(t), c_{22}^h(t)) = (y/2, 3y/8, 3y/4)$.

Thus, this economy has an equilibrium in which the money supply and the price of money are unchanging from $t = 1$ on. Each young person consumes $y/2$ when young. Per capita saving, $y/2$, is composed of contingent claims on second-period consumption supported

by a per capita portfolio consisting of real money balances equal

to y/4 and storage of the consumption good equal to y/4.

I will refer to this example and to closely related examples

below.

3. The Irrelevance Proposition.

The proposition to be proved is as follows:

If $\{\bar{c}(t-1),\ \bar{s}(t),\ \bar{K}(t),\ \bar{p}(t),\ \bar{M}(t)\}$ is an equilibrium with $\bar{p}(t) > 0$ for all $t \geq 1$ for the policy $\{G(t),\ w(t),\ K^g(t)\} = \{\bar{G}(t),\ \bar{w}(t),\ 0\}$, then $\{\bar{c}(t-1),\ \bar{s}(t),\ \bar{K}(t),\ \bar{p}(t),\ \hat{M}(t)\}$ is an equilibrium for the policy $\{\bar{G}(t),\ \hat{w}(t),\ \hat{K}^g(t)\}$, where $\{\hat{K}^g(t)\}$ is any nonnegative sequence bounded by $\{\bar{K}(t)\}$ and $\{\hat{w}(t)\}$ is any $w(t)$ sequence that for all $t \geq 1$ satisfies

(a) $\hat{w}_1{}^h(t) + \bar{s}(t)\hat{w}_2{}^h(t) = \bar{w}_1{}^h(t) + \bar{s}(t)\bar{w}_2{}^h(t)$ for each h,

and

(b) $\Sigma_h[\hat{w}_{2i}{}^h(t) - \bar{w}_{2i}{}^h(t)] = \hat{K}^g(t)[x_i - \bar{p}_i(t+1)/\bar{p}(t)]$ for each i.

(The notation "$\{\cdot(t)\}$" means a sequence defined for all $t \geq 1$.)

Before giving a proof, it is worth noting that the proposition is not vacuous.

Nonvacuousness requires that there exist economies having equilibria with $p(t) > 0$ for all t and $\bar{K}(t) > 0$ for at least some t when $K^g = 0$. The example given at the end of section two meets this requirement.

Nonvacuousness also requires the existence of at least one $\{\hat{w}(t)\}$ that satisfies (a) and (b). One such sequence is given by

$$\tilde{w}_1{}^h(t) = \bar{w}_1{}^h(t);\ \tilde{w}_{2i}{}^h - \bar{w}_{2i}{}^h(t) = \hat{K}^g(t)[x_i - \bar{p}_i(t+1)/\bar{p}(t)]/N(t)$$

for all h, i and $t \geq 1$. This endowment scheme obviously satisfies (b). To show that it satisfies (a), note that

$$\bar{s}_i(t)[\tilde{w}_{2i}(t)-\bar{w}_{2i}(t)] = \hat{K}^g(t)[\bar{s}_i(t)x_i-\bar{s}_i(t)\bar{p}_i(t+1)/\bar{p}(t)]/N(t)$$

Summing both sides over i we obtain

$$\bar{s}(t)[\tilde{w}_2^h(t) - \bar{w}_2^h(t)] = \hat{K}^g(t)[\bar{s}(t)x - 1]/N(t) = 0$$

where the first equality follows from (4) and the second from the fact that $\hat{K}^g(t) > 0$ implies $\bar{K}(t) > 0$ and, hence, (3) at equality. This and $\tilde{w}_1^h(t) = \bar{w}_1^h(t)$ imply that $\{w(t)\}$ satisfies condition (a).

Proof   By condition (a), if $\bar{c}(t)$, $\bar{s}(t)$ and $\bar{w}(t)$ satisfy (1) at equality and (2), then so do $\bar{c}(t)$, $\bar{s}(t)$ and $\hat{w}(t)$. Moreover, since (3) and (4) hold at equality at the prices $\bar{s}(t)$ and $\bar{p}(t)$ all that remains is to show that (7) is satisfied by the $\hat{M}(t)$ implied by (5) with $p(t) = \bar{p}(t)$.

We first note that

(8) $\Sigma_h[\hat{w}_1^h(t) - \bar{w}_1^h(t)] = 0$ for all $t \geq 1$

To derive this, multiply (b) by $\bar{s}_i(t)$ and sum over i to get, by way of (3) and (4) at equality, $\bar{s}(t)\{\Sigma_h[\hat{w}_2^h(t) - \bar{w}_2^h(t)]\} = 0$. This and (a) summed over h imply (8).

Now, in order to find $\hat{M}(t)$, subtract (5) for the $K^g(t) = 0$ policy from (5) for the $K^g(t) = \hat{K}^g(t)$ policy to get

(9) $\hat{K}^g(t) = \hat{T}(t)-\bar{T}(t)+[\hat{K}^g(t-1)-\bar{K}^g(t-1)]x(t)+\bar{p}(t)[\hat{M}(t)-\hat{M}(t-1)-\bar{M}(t)+\bar{M}(t-1)]$

Since $M(0)$, $K^g(0)$ and $w_{2i}(0)$ are fixed by initial conditions, (8) implies $\hat{T}(1) = \bar{T}(1)$. Thus, for $t=1$, (9) reduces to

(10) $\hat{K}^g(t) = \bar{p}(t)[\hat{M}(t) - \bar{M}(t)]$

We now show by induction that (10) holds for all $t \geq 1$. If (10) holds

for some $\bar{t} \geq 1$, then (9) for $t = \bar{t}+1$ is

(11) $\quad \hat{K}^g(\bar{t}+1) = \hat{T}(t+1) - \bar{T}(t+1) + \hat{K}^g(\bar{t})x(\bar{t}+1) + \bar{p}(\bar{t}+1)[\hat{M}(\bar{t}+1) - \bar{M}(\bar{t}+1)]$

$$- \hat{K}^g(\bar{t})\bar{p}(\bar{t}+1)/\bar{p}(\bar{t})$$

But, for all t

$$\hat{T}(t+1) - \bar{T}(t+1) = \Sigma_h[\bar{w}_{2i}^h(t) - \hat{w}_{2i}^h(t)] = \hat{K}^g(t)[x(t+1) - \bar{p}(t+1)/\bar{p}(t)]$$

where the first equality follows from (6) and (8) and the second from

condition (b). Upon substituting this into (11), we get (10) for $t = \bar{t}+1$

as required.

Now, I will use (10) to show that $\bar{c}(t-1)$, $\hat{w}(t)$ and $\hat{M}(t)$ satisfy (7).

I begin with (7) for the "–" equilibrium, namely

(12) $\qquad\qquad \Sigma_h\bar{c}_{2i}^h(t) - \Sigma_h\bar{w}_{2i}^h(t) = \bar{K}(t)x_i + \bar{p}_i(t+1)\bar{M}(t)$

Upon substituting for $\Sigma_h\bar{w}_{2i}^h(t)$ from condition (b), we have

$$\Sigma_h\bar{c}_{2i}^h(t) - \Sigma_h\hat{w}_{2i}(t) = [\bar{K}(t) - \hat{K}^g(t)]x_i + \bar{p}_i(t+1)[\bar{M}(t) + \hat{K}^g(t)/\bar{p}(t)]$$

Finally, using (10), we get

(13) $\quad \Sigma_h\bar{c}_{2i}^h(t) - \Sigma_h\hat{w}_{2i}(t) = [\bar{K}(t) - \hat{K}^g(t)]x_i + \bar{p}_i(t+1)\hat{M}(t)$

This is (7) for the asserted equilibrium under the $\hat{K}^g(t)$ policy and

completes the proof.

An Unchanged Fiscal Policy

What is the relationship between the assumptions about fiscal policy used to prove irrelevance and assumptions that hold fiscal policy unchanged, where unchanged fiscal policy means (i) an unchanged path of government consumption, (ii) an unchanged distribution of income, and (iii) an unchanged path of total taxes-minus-transfers? (Given (i), (iii) is equivalent to holding the path of the deficit unchanged.) Since (i) is an irrelevance assumption and (ii) has its counterpart in condition (a), it remains to explore the relationship between (iii) and condition (b).

If different paths of the government's portfolio imply different paths of net interest received by the government, then (iii) requires that these net interest differences be offset by differences in other components of taxes-minus-transfers. (Net interest received by government is a component of taxes-minus-transfers.) In order to express this requirement, I need a definition of time t net interest or earnings on a government portfolio $(K^g(t-1), M(t-1))$ held from $t-1$ to t. I define time t earnings on this portolio to be $[x(t) - 1]K^g(t-1) - [p(t) - p(t-1)]M(t-1)$, a definition which includes the capital loss on government liabilities in the form of money. In terms of this definition, requirement (iii) for the two portfolios $(\hat{K}^g(t-1), \bar{M}(t-1))$ and $(\hat{K}^g(t-1), \hat{M}(t-1))$ at the prices $\bar{p}(t)$ is

$$(14) \quad \bar{T}(t) - \hat{T}(t) = [x(t)-1][\hat{\bar{K}}^g(t-1)-\hat{\bar{K}}^g(t-1)] - [\bar{p}(t)-\bar{p}(t-1)][\hat{M}(t-1)-\bar{M}(t-1)]$$

where the RHS is the difference in earnings (net interest) implied by the two portfolios.

As a preliminary step, it is convenient to show that (14) and
(10) are equivalent. To show that (14) implies (10), note that (10)
for $t = 1$ follows from (9) for $t = 1$, given initial conditions
which by (14) imply $\bar{T}(1) = \hat{T}(1)$. The following induction argument
establishes (10) for all t. Assume (10) holds for some $\bar{t} \geq 1$.
Start with (9) for $t = \bar{t}+1$ and substitute into it (14) for
$t = \bar{t}+1$ and (10) for $t = \bar{t}$. The result is (10) for $t = \bar{t}+1$.
To show that (10) implies (14), simply substitute (10) for t and
$t-1$ into (9). This works for $t \geq 2$. For $t = 1$ use initial
conditions in place (10) lagged. This equivalence is not surprising
since (10) expresses the requirement that the path of net government
indebtedness at the prices $\bar{p}(t)$ be the same under the alternative
portfolios, another way of saying that the path of the deficit be the
same.

We can now establish some conclusions. Since (10) was shown to
be a consequence of the assumptions used to prove irrelevance, it
follows that those assumptions imply an unchanged path of fiscal
policy in the sense of (i) – (iii). I have not, however, been able
to establish whether the converse holds, or, in particular, whether
(14) could replace condition (b) among the assumptions used to prove
irrelevance. It is trivial to show that condition (b) is necessary
in the sense that (14) and irrelevance imply (b).$\underline{^{2/}}$ But this necessity
may be vacuous because I have not been able to determine whether (b)
is implied by (14) and the other assumptions of the irrelevance
proposition. It can, however, be shown that no "simple" endowment
scheme satisfies (14) and condition (a), but not (b) at prices
satisfying (3) and (4) at equality.

Finally, it is worth noting that condition (b) is reminiscent of a condition that holds automatically in expositons of the Modigliani-Miller theorem for corporate liability structures. Condition (b) requires that differences in earnings at t implied by different government portfolios held from $t-1$ to t be paid out in the form of taxes to agents who were present at $t-1$. For alternative corporate liability structures, this requirement is met automatically because post-state payouts by the corporation necessarily go to individuals who bought pre-state titles to those payouts. Although holding fiscal policy unchanged in the sense of (i) - (iii) guarantees that differences in earnings are paid out through the tax system, given the overlapping-generations structure, (i) - (iii) alone may not guarantee that post-state payoffs go to individuals who were present prestate (see (14) and (6)).

4. Binding Nonnegativity of Private Storage

The bound on $K^g(t)$ is necessary in order to get irrelevance. If $K^g(t) > \bar{K}(t)$ for some t, then no feasible value of $K^p(t)$ consistent with unchanged total accumulation at t exists. Irrelevance cannot, then, hold because total resources at $t+1$ in each state depend on $K^g(t)$.

While this suffices to establish necessity of the bound on $K^g(t)$, I want to display an example which suggests that some features of the usually asserted effects of open-market operations are consistent with assuming that such operations occur in a range that violates the $K^g(t) \leq \bar{K}(t)$ bound. In particular, I will display an example in which $K^g(t) > \bar{K}(t)$ amounts to a subsidy on storage financed by lump-sum taxes with the subsidy being greater and the price of money lower the greater is K^g.

The example is that given at the end of section two except that I now assume $K^g(t)/Y(t) = \theta$, $w_{2i}^h(t) = \theta y[x(t+1)-1]$ for all h and $t \geq 1$ and $\Sigma_h w_{2i}^h(0) = K^g(0)x(1)$. For each θ in $[0, 1/2]$, there is a stationary equilibrium with $p(t) = p_\theta > 0$ for all $t \geq 1$. [3/] For $\theta \leq 1/4$, the irrelevance proposition holds. For $\theta > 1/4$, the stationary solution is found by first solving the relevant versions of (1), (2), (4), and (7) with $K^p(t) = 0$ for c_1^h, c_{21}^h, c_{22}^h, s_1, s_2 and $(p_\theta M_\theta)$. [4/] Then p_θ may be found using the relevant version of (5); namely,

(15) $\theta y = p_\theta M_\theta / N - (\bar{p}\bar{M}/N)(p_\theta/\bar{p}) = p_\theta M_\theta / N - (y/4)(p_\theta/\bar{p})$

where $\bar{p}$ and $\bar{M}$ are the equilibrium values for $K^g = 0$. Without

displaying the numerical solutions, we can show that $p_\theta/\bar{p} < 1$

for some θ.

In a stationary equilibrium for this economy

(16) $y = c_1(t) + K^g/N + (\bar{p}\bar{M}/N)(p_\theta/\bar{p}) = c_1(t) + \theta y + (y/4)(p_\theta/\bar{p})$

which simply describes the disposition of the per capita endowment of

the young at $t = 1$, $p_\theta \bar{M}/N$ being the amount that goes to the current

old. For this example, $c_1(t)$ is equal to half of wealth, which by

(1) and (4) implies

(17) $c_1(t) = y - \theta y(1 - sx) \geq y - \theta y/2$

The inequality follows from noting that sx is a minimum at $s_1 = 1$

for s satisfying (4). This inequality and (16) imply $p_\theta/\bar{p} \leq (2 - 3/\theta)$

or $p_\theta/\bar{p} < 1$ for $\theta > 1/3$.

An alternative way to generate stationary equilibria with $\theta > 1/4$

is to treat $(p_\theta/\bar{p})$ as a policy instrument; the interpretation is that

the government announces a price of money, p_θ, satisfying $0 < p_\theta/\bar{p} < 1$

at which it is willing to sell (or buy) money in exchange for the

consumption good at any time. The equilibrium is found by solving (15)

and the relevant versions of (1), (2), (4), (7) and $K^p(t) = 0$ for

θ, c_1^h, c_{21}^h, c_{22}^h, s_1, s_2 and $p_\theta M_\theta$.

There is, of course, nothing "neutral" about alternative values

of $p_\theta/\bar{p}$ accomplished in either of these equivalent ways. The

example shares features of an open-market operation consisting, say,

of government purchases of mortgages on new construction in some

locality at a higher-than-market price. In conformity with the example, this stimulates construction in the locality and, hence, is not neutral. As is widely recognized, though, to call this monetary policy is stretching matters. First, fiscal policy cannot be held fixed in the face of this policy. Second, an equivalent policy is an interest subsidy which everyone agrees is fiscal policy.

5. Globally Binding Legal Minimum Money Holdings

The model described in section 2 is one of voluntarily-held money, equation (4) being a consequence. In fact, equation (4) is a consequence if _some_ money is held voluntarily. But it is easy to construct a model and there may be historical instances in which money is held _only_ to meet prescribed legal restrictions. In such situations, money can have value in an equilibrium with the LHS of (4) less than the RHS and irrelevance need not hold.

I will illustrate the nonirrelevance possibility by way of an example with a "reserve requirement": storage of k units of the consumption good from t to $t+1$ must be accompanied by storage of money from t to $t+1$ with value at t at least equal to ρk for $\rho \geq 0$. The physical environment of the example is $N(t) = 1$ and $Y(t) = y > 0$ for all t, $u^h(c_1, c_2) = u(c_1, c_2)$ with c_1 and c_2 being normal goods, and $x = (x_1) = \bar{x} > 1$. Policy is given by $G(t) = 0$, $K^g(t) = K^g$, $w_1^h(t) = y$ and $w_2^h(t) = K^g(\bar{x}-1)$ for all $t \geq 1$, and $\Sigma_h w_{2i}^h(0) = K^g(0)x(1)$. I will describe the dependence of the stationary equilibrium on the parameter K^g.

At any price s, profit from storing k units of the consumption good from t to $t+1$ consists of the profit from storing the good and the profit from storing the required money; namely, $(s\bar{x} - 1)k + [sp(t+1)/p(t) - 1]\rho k$. It follows that at $p(t+1) = p(t) = p > 0$, $s \leq (1+\rho)/(\bar{x}+\rho) < 1$ in any competitive equilibrium. It also follows that no additional money is stored at any $p(t+1) = p(t) = p > 0$. That being so, the relevant version of (7) implies $c_2 = (\bar{x}+\rho)K^p + K^g(\bar{x}-1)$. This and (1) imply $c_1 = y - (1+\rho)K^p$. Then, letting

$v(c_1, c_2)$ denote the function $u_1(c_1, c_2)/u_2(c_1, c_2)$, we may summarize (1) - (4) and (7) by

(18) $v[y - (1+\rho)K^P, (\bar{x}+\rho)K^P + K^g(\bar{x}-1)] = (\bar{x}+\rho)/(1+\rho)$ if $K^P > 0$.

A second condition on K^P and K^g (and p) is the relevant version of (5),

(19) $K^g = \rho K^P - p\bar{M}$

Here $\bar{M}$ is what the money supply would be if $K^g = 0$. Since $p\bar{M} \geq 0$, this implies $K^g \leq \rho K^P$.

Our first task is to find the pairs (K^g, K^P) that satisfy this inequality and (18). The bold faced curve in Figure 1 constitutes this set.[5/] It follows that for any K^g in $[0, K^*)$, there exists a stationary equilibrium; find K^P from (18) and, then, p from (19). It is immediate that p is decreasing K^g.

In this example and that of the last section, open market operations have the usually asserted qualitative effects on the price of money. This implies that the welfare of the current old (at t=1) is affected in a similar way by such operations. But there the similarity ends. In section 4, $K^g > \bar{K}$ implies net taxes on the young and, in simple examples, makes everyone worse off than they are with $K^g \leq \bar{K}$. In this section, $K^g > 0$ implies a net subsidy to the young and makes them better off than with $K^g = 0$. Note that in both cases, unchanged fiscal policy is not consistent with different government portfolios. Note also that open market operations seem to be consistent with "neutrality" in the sense of an unchanged real equilibrium only when the irrelevance proposition holds. When it holds, "neutrality" is accompanied by an unchanged price of money.

6. Government Bonds and Private Versus Government Intermediation Costs

So far I have described open-market operations in (titles to) real capital. In the United States, open-market operations are largely conducted in government bonds. For this discussion, I assume that such bonds take the form of default-free, zero coupon titles to fiat money in the future (discount bonds).

In order for open-market operations in such bonds to matter, it is necessary that these bonds not always sell at face value. But getting coexistence of voluntarily-held fiat money and interest-bearing bonds is not easy.[6/] Consider a bond which at time t is a title to one dollar at time t+k. At time t+k-1, it and one dollar are both titles to one dollar at t+k. Hence, if both are held, then the bond must sell for one dollar at t+k-1. By induction, then, the bond must sell for one dollar at time t. It seems evident that to avoid this one must somehow place barriers in the way of trading in bonds.

The necessity to restrict bond trading is, of course, one of the messages of the inventory models of money demand (Baumol (1952), Tobin (1956), Miller-Orr (1966)). In those models, individuals and firms require money in order to make purchases. Why, though, cannot government bonds be "spent"? One answer is that bonds can not be "spent" because they are available only in large, inconveniently sized denominations.

To convince yourself that this indivisibility is the only thing that makes bonds different from currency, consider the following hypothetical situations. Suppose the Federal Reserve stood ready to convert on demand large denomination Treasury Bills into small

denomination bills that are equivalent in terms of face value and maturity date. Would Treasury Bills sell at a discount in such circumstances? Alternatively, imagine that the Federal Reserve ceased issuing currency in anything but thousand-dollar denominations. Absent a prohibition, there would presumably appear private sector, one-hundred-percent-reserve intermediaries who, on the model of mutual funds, would issue smaller denominations. In such circumstances, the thousand-dollar bill would sell at a discount in terms of smaller denomination intermediary liabilities.[7] Indeed, this situation could be approximated if the Federal Reserve were to charge for new currency in a way that reflects its costs, a proposal now under consideration.

If government bonds sell at a discount only because they are issued in large denominations that on the margin at least have to be intermediated by the private sector using a costly technology, then one is immediately led to ask whether the government should ever issue such things. If government resource costs do not depend on the composition of its liabilities, then, price discrimination considerations aside, it should not (see Bryant and Wallace (1979 a, b)). Indeed, this answer is virtually implied by the inventory models of money demand. An increase in bonds and a decrease in money in those models is accompanied by an increase in the yield on bonds sufficient to induce additional trips-to-the-bank, additional phone calls to the broker, and so on. The higher interest on bonds must be financed by higher taxes, the effect being a higher subsidy on trips-to-the-bank financed by higher taxes.

This particular asymmetry -- namely, bonds impose resource costs on the public but do not allow the government any cost savings -- is, however, far-fetched.[8/] Klein (1973) argued that any issuer is induced to issue large denomination securities only because issuing smaller denominations would involve additional resource costs. Certainly, Federal Reserve Bank currency handling departments use up resources. Indeed, sufficient symmetry would give rise to the irrelevance result; that is, it is possible that more government bonds and less currency outstanding implies no more than a shift of intermediation activities from the government to the private sector with the additional interest cost to the government being matched by reduced government consumption in the form of reduced resource expenditures on processing currency.

Whether or not such symmetry holds, the analysis of open-market operations in government bonds in models that are consistent with the coexistence of voluntarily held valued money and interest bearing bonds is very different from the analysis of open-market operations in typical macroeconomic models. While the latter pay lip service to the inventory models of money demand, their results seem suspiciously like some of those that come from a model with a globally binding legal restriction on minimum money holdings. For example, a careful drawing out of the implications of the inventory models of money demand has to recognize that accompanying alterations in the ratio of bonds to money must be alterations in the amount of resources expended on trips--to-the-bank. Unless symmetry holds, this, in turn, alters the amount of output available for consumption and investment. If symmetry holds, we get irrelevance. In neither case do we get the qualitative answers of typical macroeconomic models.

Concluding Remarks

Most economists are aware of considerable evidence showing that the price level and the amount of money are closely related. That evidence, though, does not imply that the irrelevance proposition is inapplicable to actual economies. The irrelevance proposition applies to asset exchanges under some conditions. Most of the historical variation in money supplies has not come about by way of asset exchanges; gold discoveries, banking panics, and government deficits and surpluses account for much of it. Nothing in the models for which the irrelevance proposition holds denies that such occurrences alter the price level in the usual way. The applicability of the irrelevance proposition can, perhaps, be judged by examining periods of exogenous asset exchanges. Two episodes that come to mind are the 1920's Federal Reserve gold sterilization program and the large purchases of government bonds by the Federal Reserve in the post World War II pre-accord period.

Perhaps the main plea to be made for the irrelevance proposition is that it, and the environments in which it holds, should serve as the starting point for analyses of government asset exchanges. This is the same plea that is made for the Modigliani-Miller theory as a theory of corporate liability structures. The applicability of complete competitive markets to open market operations seems no more far-fetched than its applicability to corporate liability structures. After all, economies of complete competitive markets are ones in which a prohibition on the institution of limited liability does not matter. This last implication seems, if anything, more far-fetched than the notion that it matters little whether or not the government stands ready to convert its large denomination liabilities into a wide range

of equivalent smaller denomination liabilities, denomination being the exogenous feature that makes currency different from government bonds.

Finally, a word of apology for the title of this paper is in order. The irrelevance proposition proved in section three is defective because it leaves completely open the question, "How broad is the class of environments for which the conclusion holds?" Indeed, since the proposition is an arbitrage proposition, it may be possible to proceed without completely specifying the environment. For example, as regards individual choice, it is enough to establish that different government portfolios are consistent with the same budget set (equation (1)) for each individual. Moreover, although my notation uses the particular age composition assumed -- two as opposed to n-period lived people -- and one good, none of this is necessary for the conclusion. Since I use all these extraneous assumptions, this paper ought to be viewed as providing only a suggestion for a general Modigliani-Miller theorem for open-market operations.

Footnotes

[1] That (5) with $T(t)$ as defined in (6) is the cash-flow constraint for the government may, perhaps, be made more obvious by noting that it follows from feasibility (at equality) and a consolidated cash-flow constraint for the private sector. Letting $C(t)$ be total private consumption at t, the first of these is

$G(t) + C(t) + K(t) = x(t)K(t-1) + Y(t)$ while the second is

$C(t) + K^p(t) + p(t)M(t) = x(t)K^p(t-1) + p(t)M(t-1) + \Sigma_h w_1^h(t) + \Sigma_h w_{2i}^h(t-1)$. Equation (5) is obtained by subtracting one of these from the other.

[2] Irrelevance implies that (12) and (13) hold. Subtract (13) from (12) and substitute into the difference $(\hat{M}(t) - \bar{M}(t))$ from (10). The result is condition (b).

[3] It is not true that $1/2$ is an upper bound on θ. An (unattainable) upper bound is $2/3$.

[4] The solution for s_1 is $[5\theta/2 - 2 + 2(1 + \theta/2 + 73\theta^2)^{\frac{1}{2}}]/6\theta$. One must, of course, verify that the (s_1, s_2) solution satisfies (3). It does and with strict inequality when $\theta > 1/4$. This in turn, implies net taxes when $\theta > 1/4$.

[5] That (18) is as pictured follows from c_1 and c_2 being normal goods which implies $v_1 < 0$, $v_2 > 0$.

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Figure 1