



Appendix for Financial Frictions and Fluctuations in Volatility

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Staff Report 538

Revised November 2018

Keywords: Uncertainty shocks; Great Recession; Labor wedge; Firm heterogeneity; Credit constraints;
Credit crunch; Firm credit spreads

JEL classification: D52, D53, E23, E24, E32, E44

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Online Appendix for Financial Frictions and Fluctuations in Volatility *

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November 27, 2018

1 Computational Algorithm

In this appendix describe the computational algorithm we use to compute our model.

1.1 Definition of equilibrium

An equilibrium consists of firms' value function $V(S, z, x)$, firms' labor choice $\ell(S, z, x)$, firms' debt choice $b(S, z, x)$, firms' default cutoff $\kappa^*(S, S', z', \ell', b')$, the borrowing limits $M(S, z)$, the bond price schedule $q(S, z, \ell', b')$, the aggregate output $Y(S)$, wage $w(S)$, and the law of motion of distribution $H(\sigma', S)$ such that

1. Given the aggregate output $Y(S)$, wage $w(S)$, and the law of motion of distribution $H(\sigma', S)$, the borrowing limits $M(S, z)$, and the bond price schedule $q(S, z, \ell', b')$, the functions of $\{V(S, z, x), \ell(S, z, x), b(S, z, x), \kappa^*(S, S', z', \ell', b')\}$ solve the firm's problem. For a state (S, z, x) with $x + M(S, z) < 0$, so that the budget set is empty, $V(S, z, x) = 0$ and otherwise

$$V(S, z, x) = \max_{\ell', b'} x + q(S, z, \ell', b')b' + \beta \sum_{\sigma'} \pi(\sigma' | \sigma) \int_{z'} \int^{\kappa^*} V(S', z', x') d\Phi(\kappa) dF(z' | z, \sigma) \quad (1)$$

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subject to

$$x + q(S, z, \ell', b')b' \geq 0, \quad (2)$$

$$M(S, z) - q(S, z, \ell', b')b' \leq F_m(S, z) \quad (3)$$

$$x' = z'Y(S)^{\frac{1}{\gamma}}(\ell')^{\frac{\alpha(\gamma-1)}{\gamma}} - w(S)\ell' - b' - \kappa \quad (4)$$

$$\kappa^*(S, S', z', \ell', b') = z'Y(S)^{\frac{1}{\gamma}}(\ell')^{\frac{\alpha(\gamma-1)}{\gamma}} - w(S)\ell' - b' + M(S', z') \quad (5)$$

$$S' = (\sigma', H(\sigma', S)) \quad (6)$$

For notational simplicity, let

$$W(S, z, \ell', b') = \beta \sum_{\sigma'} \pi(\sigma'|\sigma) \int_{z'} \int^{\kappa^*} V(S', z', x') d\Phi(\kappa) dF(z'|z, \sigma)$$

and write (1) equivalently as

$$V(S, z, x) = \max_{\ell', b'} x + q(S, z, \ell', b')b' + W(S, z, \ell', b') \quad (7)$$

subject to (2)–(6).

2. The bond price schedule ensures that lenders break even,

$$q(S, z, \ell', b') = \beta \sum_{\sigma'} \pi(\sigma'|\sigma) \int_{z'} \Phi(\kappa^*(S, S', z', \ell', b')) dF(z'|z, \sigma) \quad (8)$$

and the borrowing limits satisfy

$$M(S, z) = \max_{\ell', b'} q(S, z, \ell', b')b'. \quad (9)$$

3. The consumer problem is to solve

$$V^H(A_t, S_{t-1}) = \max_{L_t} \left\{ \sum_{\sigma_t} \pi_{\sigma}(\sigma_t|\sigma_{t-1}) \max_{C_t(\sigma_t), \{A_{t+1}(\sigma_{t+1})\}} \left[\frac{C_t(\sigma_t)^{1-\sigma}}{1-\sigma} - \frac{L_t^{1+\nu}}{1+\nu} + \beta V^H(S_t) \right] \right\} \quad (10)$$

subject to the budget constraint, for each σ_t ,

$$C_t(\sigma_t) + \sum_{\sigma_{t+1}} Q(\sigma_{t+1}|\sigma_t) A_{t+1}(\sigma_{t+1}, \sigma_t) = W_t(S_{t-1})L_t + A_t(\sigma_t) + D_t(\sigma_t, S_{t-1}),$$

where A_t is the vector of assets $\{A_t(\sigma_t)\}$. The first-order condition for the consumption–labor choice is

$$-\frac{\sum_{\sigma_t} \pi_{\sigma}(\sigma_t|\sigma_{t-1}) U_L(C_t(\sigma_t), L_t)}{\sum_{\sigma_t} \pi_{\sigma}(\sigma_t|\sigma_{t-1}) U_C(C_t(\sigma_t), L_t)} = W_t(S_{t-1}). \quad (11)$$

Using the envelope condition and $Q(\sigma_{t+1}|\sigma_t) = \beta\pi(\sigma_{t+1}|\sigma)$, and using the additive separability of utility, the first-order condition for consumption implies $C_t(\sigma_t) = C_{t+1}(\sigma_{t+1})$ all σ_{t+1} , so that consumption is constant, say $C_t(\sigma_t) = \bar{C}$ for all t . Using the functional form for utility, the first-order condition (11) reduces to

$$\bar{C}^\sigma L_t(\sigma_{t-1})^v = W_t(S_{t-1}). \quad (12)$$

4. Aggregate wages and the face value of wages are related by $W(S_{t-1})L(S_{t-1}) =$

$$\begin{aligned} W(S_{t-1})L(S_{t-1}) = & \\ & w(S_{t-1}) \int \pi_\sigma(\sigma_t|\sigma_{t-1})\pi(z_t|z_{t-1}, \sigma_{t-1}) \int_{\kappa \in \Omega_R} d\Phi(\kappa) \ell_t \Upsilon(z_{t-1}, x_{t-1}) + \\ & \int \pi_\sigma(\sigma_t|\sigma_{t-1})\pi(z_t|z_{t-1}, \sigma_{t-1}) \int_{\kappa \in \Omega_D} \max\{p_t \ell_t^\alpha - w_{mt} - \kappa, 0\} d\Phi(\kappa) \Upsilon(z_{t-1}, x_{t-1}). \end{aligned}$$

5. The market clearing conditions for labor and output are

$$\begin{aligned} L(S) &= \int \ell(S, z, x) \Upsilon(z, x) \\ Y(S) &= \left[\int z \ell^{\alpha \frac{\gamma-1}{\gamma}}(S, z, x) \Upsilon(z, x) \right]^{\frac{\gamma}{\gamma-1}}. \end{aligned} \quad (13)$$

6. The law of motion of distribution is consistent with the policy functions of firms, households, and shocks.

1.2 Algorithm overview

In order to solve the individual firm's optimization problem, the firm needs to forecast next period's wage $w(S)$ and next period's output $Y(S)$, and it needs a transition law for the aggregate state. In practice, it is infeasible to include the entire measure Υ in the state. Instead, we follow a version of Krusell and Smith (1998) to approximate the forecasting rules for the firm. We do so by approximating the distribution of firms Υ with lags of aggregate shocks, $(\sigma_{-1}, \sigma_{-2}, \sigma_{-3}, k)$ where k records how many periods the aggregate shocks have been unchanged. Here $k = 1, \dots, \bar{k}$ and \bar{k} is the upper bound on this number of periods. We set $\bar{k} = 9$. In a slight abuse of notation, we use $S = (\sigma, \sigma_{-1}, \sigma_{-2}, \sigma_{-3}, k)$ in the rest of this description of the algorithm to denote our approximation to the aggregate state. The law of motion of (our approximation to) the aggregate state is given by $H(\sigma', S) = (\sigma', \sigma, \sigma_{-1}, \sigma_{-2}, k')$ with $k' = k + 1$ if $\sigma' = \sigma = \sigma_{-1} = \sigma_{-2}$ and 0 otherwise. Given our parameterization for $\sigma = \{\sigma_L, \sigma_H\}$ and $\bar{k} = 9$, the total number of points for the mutually exclusive aggregate states S is 32.

We start with an initial guess of two arrays for the aggregate wages and output, $w^0(S)$ and $Y^0(S)$, referred to as *aggregate rules*. We then solve the model with two loops: an inner and an

outer loop. In the inner loop, taking as given the current set of aggregate rules, we iteratively solve each firm's optimization problem until convergence. In the outer loop, taking as given the converged decisions from the inner loop, we start with a distribution of firms $\Upsilon_0(z, x)$ and simulate the economy for T periods. In each period t , we record firms' labor choice $\{\ell_{t+1}(z, x)\}$, borrowing $\{b_{t+1}(z, x)\}$, and default decisions $\{\iota_t(z, x)\}$. Moreover, we use (12) and (13) to construct new guesses $w_{t+1}(S)$ and $Y_{t+1}(S)$ for the aggregate rules. We then repeat the procedure until the arrays of aggregate output and wages converge.

1.2.1 Inner loop

Before we solve the inner loop, we discretize the idiosyncratic productivity shock $z(\sigma_{-1})$ using the Gaussian quadrature method. The discretization of this shock consists of 12 productivity points for each level of volatility σ_{-1} and transition matrices $\pi_z(z'|\sigma_{-1}, \sigma)$ that depend on σ_{-1} and σ . The idiosyncratic state x is discretized into 15 endogenous grids that depend on the shocks z and the aggregate state S . The state space for the firm's problem has $\#S \times \#Z \times \#X = 5,760$ grid points. We also discretize the revenue shock κ into 100 points using the Gaussian quadrature method and use it to evaluate the integrals in the firm's future value.

In the loop, taking as given the current set of aggregate rules, say, $w(S) = w^k(S)$ and $Y(S) = Y^k(S)$, we first construct the bond price schedule and borrowing limits recursively. We then solve firms' decision rules.

Borrowing Limits We start with an initial guess for the borrowing limits $M^0(S, z)$ that is looser than the actual borrowing limit. We set grids for $\{\ell', b'\}$, with 32 points for ℓ' and 64 points for b' . The grid for ℓ' is set around the frictionless choice of labor and depends on $\{S, z\}$. The grid for b' is endogenous and depends on $\{S, z, \ell'\}$. We update the grid on b' with every iteration of the borrowing limit. With these choices, the resulting array for $q(S, z, \ell', b')$ has $\#L \times \#B \times \#Z \times \#S = 786,432$ grid points.

Given $M^0(S, z)$, we construct the associated default cutoff

$$\kappa^{*0}(S, S', z', \ell', b') = z'Y(S)^{\frac{1}{\gamma}}(\ell')^{\frac{\alpha(\gamma-1)}{\gamma}} - w(S)\ell' - b' + M^0(S', z') \quad (14)$$

and the associated bond price schedule

$$q^0(S, z, \ell', b') = \beta \sum_{\sigma'} \pi(\sigma'|\sigma) \int_{z'} \Phi[\kappa^{*0}(S, S', z', \ell', b')] dF(z'|\sigma, z). \quad (15)$$

In the first step of the iteration, we update the borrowing limit to $M^1(S, z)$ using

$$M^1(S, z) = \max_{\ell', b'} q^0(S, z, \ell', b')b' \quad \text{for each } (S, z)$$

and then construct the associated default cutoff array $\{\kappa^{*1}(S, S', z', \ell', b')\}$ and bond price schedule

array $\{q^1(S, z, \ell', b')\}$ using the analogs of (14) and (15).

We continue this process iteratively until the constructed sequence of borrowing limit arrays $\{M^n(S, z)\}$ converge. We then record the associated arrays of default cutoffs and bond price schedules, denoted $\{\kappa^*(S, S', z', \ell', b')\}$ and $\{q(S, z, \ell', b')\}$ which we hold fixed during each iteration of the firm decision rules that we describe next.

Firm Decision Rules Given the converged borrowing limits and associated default cutoffs and bond price schedule, we solve for the firms' decision rules iterating over a combination of policy functions and value functions. For each grid point, we solve a system of two nonlinear equations by interpolating over the policy functions using a multivariate finite element method. We use the Intel Fortran compiler using the IMSL routine DNEQNF.

Let $\gamma(S, z, x)$ and $\mu(S, z, x)$ denote the multipliers on the nonnegative equity payout condition (2), denoted NEP , and the manager deviation condition (3), denoted MD , respectively. The following first-order conditions characterize firms' optimization problem (1):

$$\begin{aligned} (1 + \gamma + \mu) \frac{\partial q(S, z, \ell', b')b'}{\partial \ell'} + \frac{\partial W(S, z, \ell', b')}{\partial \ell'} &= 0 \\ (1 + \gamma + \mu) \frac{\partial q(S, z, \ell', b')b'}{\partial b'} + \frac{\partial W(S, z, \ell', b')}{\partial b'} &= 0, \end{aligned} \tag{16}$$

where

$$\begin{aligned} \frac{\partial q(S, z, \ell', b')b'}{\partial \ell'} &= \beta b' \sum_{\sigma'} \pi(\sigma', \sigma) \left[\sum_{i=1}^{N_z} \pi_z(z_i | z, \sigma) \phi(\kappa^*(S, S', z_i, \ell', b')) [\alpha_h z_i Y^{\frac{1}{\gamma}}(S) (\ell')^{\alpha_h - 1} - w(S)] \right] \\ \frac{\partial q(S, z, \ell', b')b'}{\partial b'} &= \beta \sum_{\sigma'} \pi(\sigma', \sigma) \left[\sum_{i=1}^n \pi_z(z_i | z, \sigma) [\Phi(\kappa^*(S, S', z_i, \ell', b')) - b' \phi(\kappa^*(S, S', z_i, \ell', b'))] \right] \end{aligned}$$

and

$$\begin{aligned}\frac{\partial W}{\partial \ell'} &= \beta \sum_{\sigma'} \pi(\sigma'|\sigma) \sum_{i=1}^{N_z} \int^{\kappa^*(S, S', z_i, \ell', b')} \left[(1 + \gamma(S', z_i, x')) \alpha_h z_i Y(S)^{\frac{1}{\gamma}} \ell^{\alpha_h - 1} \right] d\Phi(\kappa) \pi_z(z_i|z, \sigma) \\ &\quad - \beta \sum_{\sigma'} \pi(\sigma'|\sigma) \sum_{i=1}^{N_z} \int^{\kappa^*(S, S', z_i, \ell', b')} [1 + \gamma(S', z_i, x')] w(S) d\Phi(\kappa) \pi_z(z_i|z, \sigma) \\ &\quad + \beta \sum_{\sigma'} \pi(\sigma'|\sigma) \sum_{i=1}^{N_z} V(x^*(\ell', b', z_i, S), z', S') \kappa_{\ell}^*(S, S', z_i, \ell', b') \pi_z(z_i|z, \sigma)\end{aligned}$$

$$\begin{aligned}\frac{\partial W}{\partial b'} &= -\beta \sum_{\sigma'} \pi(\sigma'|\sigma) \sum_{i=1}^{N_z} \int_0^{\kappa^*(S, S', z_i, \ell', b')} [1 + \gamma(S', z_i, x')] d\Phi(\kappa) \pi_z(z_i|z, \sigma) \\ &\quad + \beta \sum_{\sigma'} \pi(\sigma'|\sigma) \sum_{i=1}^{N_z} V(x^*(\ell', b', z_i, S), z_i, S') \kappa_b^*(S, S', z_i, \ell', b') \pi_z(z_i|z, \sigma),\end{aligned}$$

where $\alpha_h = \alpha \frac{(\eta-1)}{\eta}$, $\kappa_{\ell}^* = \frac{\partial \kappa^*}{\partial \ell'}$, and $\kappa_b^* = \frac{\partial \kappa^*}{\partial b'}$.

In the iterations to solve for the firm decision rules, we iterate on a set of arrays of grids $\{X(S, z)\}$ where

$$X(S, z) = \{x_1, \dots, x_N\},$$

where the set of points $\{x_1, \dots, x_N\}$ varies with (S, z) . Let $\{X^0(S, z)\}$ denote the initial guess on the array of grids. We also begin with an initial guess for the multiplier function $\{\gamma^0(S, z, x)\}$ on the NEP condition and the value function $\{V^0(S, z, x)\}$. Both the multiplier functions and the value functions are defined not just on the grid but also for all values of x in a range $[-M(S, z), \infty]$ as we interpolate between the grid points.

For each iteration n , given the array of grids $\{X^n(S, z)\}$, the multipliers $\{\gamma^n(S, z, x)\}$ and the value function $\{V^n(S, z, x)\}$ from the previous iteration, we solve for the updated array of grids $\{X^{n+1}(S, z)\}$, the multiplier function $\{\gamma^{n+1}(S, z, x)\}$ and the value function $\{V^{n+1}(S, z, x)\}$ in two steps. In these steps, we use the fact that for all cash-on-hand levels x greater than some cutoff level $\hat{x}(S, z)$ the NEP is not binding and the decision rules for labor $\ell'(S, z, x)$ and debt $b'(S, z, x)$ do not vary with x . We refer to the associated values of labor and debt as the *nonbinding levels of labor and debt* and denote them by $\hat{\ell}(S, z)$ and $\hat{b}(S, z)$. So given $\gamma^n(S, z, x)$ and the value function $\{V^n(S, z, x)\}$, we proceed as follows.

1. Solve for the cutoff $\hat{x}(S, z)$ by solving for the values $\hat{\ell}(S, z)$ and $\hat{b}(S, z)$. To do so, we impose that the NEP condition is not binding.
 - (a) Assume the manager deviation condition MD is also not binding and solve for the *tentative* solutions $\ell'_{tent}(S, z)$ and $b'_{tent}(S, z)$. The tentative solutions solve the following two

equations in ℓ' and b' :

$$\begin{aligned}\frac{\partial q(S, z, \ell', b')b'}{\partial \ell'} + \frac{\partial W(S, z, \ell', b')}{\partial \ell'} &= 0 \\ \frac{\partial q(S, z, \ell', b')b'}{\partial b'} + \frac{\partial W(S, z, \ell', b')}{\partial b'} &= 0\end{aligned}$$

We then check whether the constructed tentative solutions satisfy the manager deviation condition. If so, then we set $\ell'_f(S, z) = \ell'_{tent}(S, z)$ and $b'_f(S, z) = b'_{tent}(S, z)$. If not, we continue to step (b).

- (b) If we reach this step, we know that the manager deviation condition is binding. We thus impose the MD condition with equality and define $\hat{\ell}(S, z)$ and $\hat{b}(S, z)$ as the solution to

$$\begin{aligned}M(S, z) - q(S, z, \ell', b')b' &= F_m(S, z) \\ \frac{\partial q(S, z, \ell', b')b'}{\partial b'} \frac{\partial W(S, z, \ell', b')}{\partial \ell'} - \frac{\partial q(S, z, \ell', b')b'}{\partial \ell'} \frac{\partial W(S, z, \ell', b')}{\partial b'} &= 0,\end{aligned}$$

where this last equation is derived by combining the two first-order conditions in (16) and eliminating the multipliers.

- (c) Construct the grid $\{X^{n+1}(S, z)\} = \{x_1, x_2, \dots, x_N\}$ by setting

$$x_1 = -M(S, z) \text{ and } x_N = \hat{x}(S, z).$$

That is, we know that if the cash-on-hand x is so low that even if the firm borrows the maximum amount $M(S, z)$, the associated dividends $d = x + M(S, z)$ is negative, the firm will default. We also know that if the cash-on-hand is sufficiently high, so that $x \geq \hat{x}(S, z)$, the optimal decisions will be given by the nonbinding levels of labor and debt $\hat{\ell}(S, z)$ and $\hat{b}(S, z)$. We then choose a set of intermediate points $\{x_2, \dots, x_{N-1}\}$.

2. Solve for decisions at the intermediate points.

We claim that at any of these intermediate points with $-M(S, z) < x < \hat{x}(S, z)$, the MD condition is not binding. To prove this claim, we note that since $x < \hat{x}(S, z)$, then $-x > -\hat{x}(S, z)$, and the firm must borrow more at x than at \hat{x} to keep dividends nonnegative. Since at \hat{x} the nonnegative equity payout condition binds, implying $d = 0$, then from $d = \hat{x} + qb'$ and the manager deviation condition, we know that

$$-\hat{x}(S, z) = q(S, z, \hat{\ell}, \hat{b})\hat{b} \geq M(S, z) - F_m(S, z),$$

so

$$-x \geq M(S, z) - F_m(S, z).$$

Thus, for each intermediate point $x \in \{x_2, \dots, x_{N-1}\}$, the NEP condition is binding and the

manager deviation condition is not binding. For each such x , we solve ℓ', b' from the two equations

$$\frac{\partial q(S, z, \ell', b') b'}{\partial b'} \frac{\partial W(S, z, \ell', b')}{\partial \ell'} - \frac{\partial q(S, z, \ell', b') b'}{\partial \ell'} \frac{\partial W(S, z, \ell', b')}{\partial b'} = 0$$

$$x + q(S, z, \ell', b') b' = 0.$$

Let the solution be $\ell'(S, z, x)$ and $b'(S, z, x)$. We then compute the new multiplier γ^{n+1} from

$$\gamma^{n+1}(S, z, x) = -\frac{\frac{\partial W}{\partial \ell'}}{\frac{\partial q(S, z, \ell', b') b'}{\partial \ell'}} - 1,$$

where the derivatives are evaluated at the solution $\ell'(S, z, x)$ and $b'(S, z, x)$.

3. We then update the value function to V^{n+1} using

$$V^{n+1}(S, z, x) = x + q(S, z, \ell', b') b'$$

$$+ \beta \sum_{\sigma} \pi(\sigma' | \sigma) \sum_i \int^{\kappa^*(S, S', z', \ell', b')} V^n(S', z', x') d\Phi(\kappa) \pi(z_i | z, \sigma),$$

where ℓ' and b' are shorthand notations for $\ell(S, z, x)$ and $b(S, z, x)$.

4. Iterate. We iterate steps 1-3 until the multipliers $\gamma^n(S, z, x)$ and the value functions $V^n(S, z, x)$ converge.

1.2.2 Outer loop

We simulate the model for T periods. For each period, the economy has an aggregate state of $S_t = (\sigma_t, \sigma_{t-1}, \sigma_{t-2}, \sigma_{t-3}, k_t)$. We set a time-varying grid of $X_t(z)$ of 80 points. The time-varying distribution of firms $\Upsilon_t(z, x)$ is an array of $12 \times 80 = 960$ points.

1. For each firm in the distribution Υ_t , we define the *default decision* $\iota_t(z, x)$ by $\iota_t(z, x) = 1$ if and only if $x \leq -M(S_t, z)$. For nondefaulting firms, we calculate their labor choice $\ell_{t+1}(z, x)$ and debt choice $b_{t+1}(z, x)$ by interpolating the decision rules $\ell(S_t, z, x)$ and $b(S_t, z, x)$ from the inner loop. Summing over default decisions of firms, we get the total mass of exiting firms

$$E_t = \sum_{z, x \in X_t(z)} \iota_t(z, x) \Upsilon_t(z, x).$$

2. We find the labor and debt of new entrants $(\ell_{t+1}^e(\omega), b_{t+1}^e(\omega))$. A new entrant can enter if it draws a sufficiently low entry cost in that

$$\omega \leq M(S_t, z^e).$$

We assume that from the measure of potential entrants with $\omega \leq M(S_t, z_e)$, a subset is chosen randomly so that the measure of entering firms equals E_t . Upon entry, the labor choice and new borrowing are given by $\ell_{t+1}^e(\omega, z) = \ell(S_t, z, -\omega)$, $b_{t+1}^e(\omega, z) = b(S_t, z, -\omega)$, respectively.

3. Next period's output is given by

$$Y_{t+1} = \left[\sum_{(z,x)} (1 - \iota_t(z, x)) z \ell_{t+1}(z, x)^{\alpha_h} \Upsilon_t(z, x) + E_t \int_{\omega \leq M(S_t, z^e)} z^e (\ell_{t+1}^e(z^e, \omega))^{\alpha_h} d\Omega(\omega) \right]^{\frac{\gamma}{\gamma-1}},$$

next period's labor is given by

$$L_{t+1} = \sum_{(z,x)} (1 - \iota_t(z, x)) \ell_{t+1}(z, x) \Upsilon_t(z, x) + E_t \int_{\omega \leq M(S_t, z^e)} \ell_{t+1}^e(z^e, \omega) d\Omega(\omega),$$

and next period's wage is given by

$$W_t(S_{t-1}) = \bar{C}^\sigma L_t(\sigma_{t-1})^v. \quad (17)$$

Since the choices of x_{t+1} vary smoothly with the shocks at t and we record only a finite number of grid points x_i for $i = 1, \dots, N$, when updating the distribution Υ we need to assign the mass for any (z', x_{t+1}) to points on the grid (z', x_i) . We do so by allocating the mass for any x_{t+1} to the two closest grid points x_{i-1} and x_i where $x_{i-1} \leq x_{t+1} \leq x_i$ in proportion to how close the point is to each. Specifically, let $\Lambda(x_i, x_{t+1})$ be the probability that the choice of x_{t+1} is assigned to x_{i-1} or x_i :

$$\Lambda(x_i, x_{t+1}) = \frac{x_{t+1} - x_{i-1}}{x_i - x_{i-1}} \text{ and } \Lambda(x_{i-1}, x_{t+1}) = 1 - \Lambda(x_i, x_{t+1}),$$

and $\Lambda(x_i, x_{t+1}) = 0$ if $x_{t+1} \notin [x_{i-1}, x_{i+1}]$. Then next period's distribution $\Upsilon_{t+1}(z', x_i)$ for x_i on $X_{t+1}(z')$ is given by

$$\Upsilon_{t+1}(z', x_i) = \sum_{x \in X_t(z), z} \left\{ \int^{\kappa_{t+1}^*(z', z, x)} \Lambda(x_i, x_{t+1}(z, x, z', \kappa')) d\Phi(\kappa') \right\} \pi(z'|z, \sigma_t) \Upsilon_t(z, x),$$

where x_{t+1} and κ_{t+1}^* are given by

$$\begin{aligned} x_{t+1}(z, x; z', \kappa') &= z' Y(S_t)^{\frac{1}{\gamma}} \ell_{t+1}(z, x)^{\alpha_h} - w(S_t) \ell_{t+1}(z', x) - b_{t+1}(z, x) - \kappa', \\ \kappa_{t+1}^*(z', z, x) &= \kappa^*(S_t, \sigma_{t+1}, S_{t+1}, z', \ell'(S_t, z, x), b'(S, z, x)). \end{aligned}$$

We finally project the simulated values for wages and output on a set of dummy variables corresponding to the state S . We use the fitted values as the new aggregate rules $w(S) = w^{k+1}(S)$ and $Y(S) = Y^{k+1}(S)$.

4. We iterate the outer loop until the aggregate rules converge.

1.3 Accuracy checks

Here we report the accuracy checks for the baseline result. We compute standard distance measures across two iterations as follows. For any array $f(\nu)$ over grid ν , we follow Judd's textbook¹ and compute the *distance* across iterations n and $n - 1$ as $\frac{(\sum_{\nu} (f^n(\nu) - f^{n-1}(\nu))^2)^{1/2}}{1 + (\sum_{\nu} f^{n-1}(\nu)^2)^{1/2}}$. The borrowing limit $M(S, z)$ distance is 10^{-8} . The distance for the stacked policy functions $\gamma(S, z, x)$ and $V(S, z, x)$ is 10^{-6} . In terms of the aggregate rules, $Y(S)$ and $w(S)$, the distance is 10^{-5} . The maximum Euler equation error in the firms' optimization problem is 10^{-8} .

¹Judd, Kenneth. 1998. *Numerical Methods in Economics*. MIT Press.